

Matrix Form of Finite Difference Schemes: Non-Periodic Wave Equation

Example 1: Linear Wave Equation, non-periodic, 11 Grid Points

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$
$$BC : u(0, t) = u_{left}$$
$$IC : u(x, 0) = g(x)$$

Second Order Central Differencing

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}(-u_{j-1} + u_{j+1}) = 0$$

Assume a vector of 10 grid points of computed data $\bar{u} = [u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}]$

Boundary condition: $u_1 = u_{left}$

Difference Equations

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}(-u_{j-1} + u_{j+1}) = 0$$

$$\begin{aligned}
 \frac{\partial u_2}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} & -\frac{a}{2\Delta x} u_1 & = 0 \\
 \frac{\partial u_3}{\partial t} + \frac{a}{2\Delta x} \{ & -u_2 & 0 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_4}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & -u_3 & 0 & u_5 & 0 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_5}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & -u_4 & 0 & u_6 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_6}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & -u_5 & 0 & u_7 & 0 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_7}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & -u_6 & 0 & u_8 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_8}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & -u_7 & 0 & u_9 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_9}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & -u_8 & 0 & u_{10} & 0 & \} & & = 0 \\
 \frac{\partial u_{10}}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_9 & 0 & u_{11} & \} & & = 0 \\
 \frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_{10} & 0 & \} & + \frac{a}{2\Delta x} u_{12} & = 0
 \end{aligned}$$

BC: $u_1 = u_{left}$

Note the u_{12} does not exist, so we need to do something.

Difference Equations, With Correct BC

Replace equation for u_{11} with the first order backward difference scheme

$$\frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x} [-2u_{10} + 2u_{11}] = 0$$

$$\begin{aligned}
 \frac{\partial u_2}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} & -\frac{a}{2\Delta x} u_{left} & = 0 \\
 \frac{\partial u_3}{\partial t} + \frac{a}{2\Delta x} \{ & -u_2 & 0 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_4}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & -u_3 & 0 & u_5 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_5}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & -u_4 & 0 & u_6 & 0 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_6}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & -u_5 & 0 & u_7 & 0 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_7}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & -u_6 & 0 & u_8 & 0 & 0 & \} & & = 0 \\
 \frac{\partial u_8}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & -u_7 & 0 & u_9 & 0 & \} & & = 0 \\
 \frac{\partial u_9}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & -u_8 & 0 & u_{10} & \} & & = 0 \\
 \frac{\partial u_{10}}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_9 & 0 & u_{11} & \} & & = 0 \\
 \frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2u_{10} & 2u_{11} & \} & & = 0
 \end{aligned}$$

Matrix Form, Second Order Central Differencing

The previous set of equations can be rewritten in a matrix form:

$$\frac{\partial \bar{u}}{\partial t} + \frac{a}{2\Delta x} [A] \bar{u} + \bar{bc} = \bar{0}$$

with

$$[A] = \begin{Bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{Bmatrix}$$

and $\bar{bc} = [-\frac{a}{2\Delta x} u_{left}, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

A Coupled System of ODE's

Matrix Notation: $A = B(10 : \bar{b}, \bar{c}, 1)$ with $\bar{b} = [-1, -1, -1, -1, -1, -1, -1, -1, -2]$ and $\bar{c} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 2]$

Matrix Form of Finite Difference Schemes: Non-Periodic Diffusion Equation

Example 2: Diffusion Equation, non-periodic, 11 Grid Points

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$
$$BC : u(0, t) = u_{left}, u(1, t) = u_{right}$$
$$IC : u(x, 0) = g(x)$$

Second Order Central Differencing

$$\frac{\partial u_j}{\partial t} = \frac{\mu}{\Delta x^2} (u_{j-1} - 2u_j + u_{j+1})$$

Assume a vector of 9 grid point of computed data $\bar{u} = [u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}]$

Boundary condition: $u_1 = u_{left}$ and $u_{11} = u_{right}$

Difference Equations

$$\frac{\partial u_j}{\partial t} = \frac{\mu}{\Delta x^2}(u_{j-1} - 2u_j + u_{j+1})$$

$$\begin{aligned}
 \frac{\partial u_2}{\partial t} &= \frac{\mu}{\Delta x^2} \{ -2u_2 \quad u_3 \quad 0 \quad \} + \frac{\mu}{\Delta x^2} u_{left} \\
 \frac{\partial u_3}{\partial t} &= \frac{\mu}{\Delta x^2} \{ u_2 \quad -2u_3 \quad u_4 \quad 0 \quad \} \\
 \frac{\partial u_4}{\partial t} &= \frac{\mu}{\Delta x^2} \{ 0 \quad u_3 \quad -2u_4 \quad u_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \} \\
 \frac{\partial u_5}{\partial t} &= \frac{\mu}{\Delta x^2} \{ 0 \quad 0 \quad u_4 \quad -2u_5 \quad u_6 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \} \\
 \frac{\partial u_6}{\partial t} &= \frac{\mu}{\Delta x^2} \{ 0 \quad 0 \quad 0 \quad u_5 \quad -2u_6 \quad u_7 \quad 0 \quad 0 \quad 0 \quad 0 \quad \} \\
 \frac{\partial u_7}{\partial t} &= \frac{\mu}{\Delta x^2} \{ 0 \quad 0 \quad 0 \quad 0 \quad u_6 \quad -2u_7 \quad u_8 \quad 0 \quad 0 \quad 0 \quad \} \\
 \frac{\partial u_8}{\partial t} &= \frac{\mu}{\Delta x^2} \{ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad u_7 \quad -2u_8 \quad u_9 \quad 0 \quad 0 \quad \} \\
 \frac{\partial u_9}{\partial t} &= \frac{\mu}{\Delta x^2} \{ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad u_8 \quad -2u_9 \quad u_{10} \quad 0 \quad \} \\
 \frac{\partial u_{10}}{\partial t} &= \frac{\mu}{\Delta x^2} \{ 0 \quad u_9 \quad -2u_{10} \quad 0 \quad \} + \frac{\mu}{\Delta x^2} u_{right}
 \end{aligned}$$

Note boundary conditions are included.

Matrix Form, Second Order Central Differencing

The previous set of equations can be rewritten in a matrix form:

$$\frac{\partial \bar{u}}{\partial t} = \frac{\mu}{\Delta x^2} [A] \bar{u} + \bar{b}$$

with

$$[A] = \begin{Bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{Bmatrix}$$

$$\text{and } \bar{b} = \left[\frac{\mu}{\Delta x^2} u_{left}, 0, 0, 0, 0, 0, 0, \frac{\mu}{\Delta x^2} u_{right} \right]$$

A Coupled System of ODE's

Matrix Notation: $A = B(9 : 1, -2, 1)$

Matrix Form of Finite Difference Schemes: Periodic Wave Equation

Example 3: Linear Wave Equation, periodic, 11 Grid Points

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$
$$BC : u(0, t) = u(2\pi, t) \quad IC : u(x, 0) = g(x)$$

Second Order Central Differencing

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}(-u_{j-1} + u_{j+1}) = 0$$

Assume a vector of 11 grid points of computed data $\bar{u} = [u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}]$

Difference Equations

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}(-u_{j-1} + u_{j+1}) = 0$$

$$\begin{aligned}
 \frac{\partial u_1}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_{11} & \} = 0 \\
 \frac{\partial u_2}{\partial t} + \frac{a}{2\Delta x} \{ & -u_1 & 0 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} = 0 \\
 \frac{\partial u_3}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & -u_2 & 0 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & \} = 0 \\
 \frac{\partial u_4}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & -u_3 & 0 & u_5 & 0 & 0 & 0 & 0 & 0 & \} = 0 \\
 \frac{\partial u_5}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & -u_4 & 0 & u_6 & 0 & 0 & 0 & 0 & \} = 0 \\
 \frac{\partial u_6}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & -u_5 & 0 & u_7 & 0 & 0 & 0 & \} = 0 \\
 \frac{\partial u_7}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & -u_6 & 0 & u_8 & 0 & 0 & \} = 0 \\
 \frac{\partial u_8}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & -u_7 & 0 & u_9 & 0 & \} = 0 \\
 \frac{\partial u_9}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_8 & 0 & u_{10} & 0 & \} = 0 \\
 \frac{\partial u_{10}}{\partial t} + \frac{a}{2\Delta x} \{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_9 & 0 & u_{11} & 0 & \} = 0 \\
 \frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x} \{ & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_{10} & 0 & \} = 0
 \end{aligned}$$

Because of the periodic BC: At $j = 1$ we need $u_{j-1} = u_0 = u_{11}$

Likewise at $j = 11$ we need $u_{j+1} = u_{12} = u_1$

These conditions are folded into the above equations

Matrix Form, Second Order Central Differencing

The previous set of equations can be rewritten in a matrix form:

$$\frac{\partial \bar{u}}{\partial t} + \frac{a}{2\Delta x} [A] \bar{u} = \bar{0}$$

with

$$[A] = \begin{Bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{Bmatrix}$$

A Coupled System of ODE's

Matrix Notation: $A = B_p(11 : -1, 0, 1)$